NOTE ON AREAS AND VOLUMES.

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There is so much interest centred around the curve $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}$ $+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}=1$, and the surface $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}+\left(\frac{z}{c}\right)^{\frac{2}{2p+1}}=1$, that it is necessary to give to the mathematicians a general formula for each that will hold for any positive integral values of m, n, p.

$$A=\text{ area }=4\int\int dxdy=\frac{\frac{4ab}{4}}{(2m+1)(2n=1)}\cdot\frac{\Gamma\left(\frac{2m+1}{2}\right)\ \Gamma\left(\frac{2n+1}{2}\right)}{\Gamma\left(\frac{2m+1}{2}+\frac{2n+1}{2}+1\right)}$$

$$=\frac{4ab}{\frac{2}{2m+1}+\frac{2}{2n+1}}\cdot\frac{\Gamma\left(\frac{2m+1}{2}\right)\ \Gamma\left(\frac{2n+1}{2}\right)}{\Gamma\left(\frac{2m+1}{2}+\frac{2n+1}{2}\right)},$$ by Dirichlet's theorem under the above conditions.

$$\therefore\ A=\frac{ab(2m+1)(2n+1)}{(m+n)(m+n+1)}\cdot\frac{\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(m+n)}\ \ldots.(1)$$

$$=\frac{1.3.5\ldots.(2m+1)\times 1.3.5\ldots.(2n+1)}{2.4.6\ldots.2(m+n+1)}\ .\ 2\pi ab.$$

When $m=n=0$, $A=\pi ab$, area of the curve $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$, the ellipse.

When $m=n=1$, $A=\frac{3}{8}\pi ab$, area of the hypocycloid, $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

When $m=0$, $n=1$, $A=\frac{3}{4}\pi ab$, area of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

When $m=n=2$, $A=\frac{15}{128}\pi ab$, area of the curve $\left(\frac{x}{a}\right)^{\frac{2}{5}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}=1$.

Vhen $m=1$, $n=2$, $A=\frac{15}{64}\pi ab$, area of curve $\left(\frac{x}{a}\right)^{\frac{2}{5}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}=1$.

When $m=n=4$, $A=\frac{1.5.7.9.\pi ab}{(32)^3}$, area of the curve $\left(\frac{x}{a}\right)^{\frac{2}{9}}+\left(\frac{y}{b}\right)^{\frac{2}{9}}=1$.

$V=$ volume $=8\int\int\int dxdydz$

$$=\frac{8abc}{\dfrac{8}{(2m+1)(2n+1)(2p+1)}}\cdot\frac{\Gamma\left(\frac{2m+1}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)\Gamma\left(\frac{2p+1}{2}\right)}{\Gamma\left(\frac{2m+1}{2}+\frac{2n+1}{2}+\frac{2p+1}{2}+1\right)}$$

$$=\frac{8abc}{\dfrac{4}{(2m+1)(2n+1)}+\dfrac{4}{(2n+1)(2p+1)}+\dfrac{4}{(2m+1)(2p+1)}}\times$$

$$\frac{\Gamma\left(\frac{2m+1}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)\Gamma\left(\frac{2p+1}{2}\right)}{\Gamma\left(\frac{2m+1}{2}+\frac{2n+1}{2}+\frac{2p+1}{2}\right)}\cdot$$

$$=\frac{4abc(2m+1)(2n+1)(2p+1)}{(2m+2n+2p+3)(2m+2n+2p+1)}\cdot\frac{\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{\Gamma(m+n+p+\frac{1}{2})}\ldots(2)$$

$$=\frac{1.3.5\ldots(2m+1)\times 1.3.5\ldots(2n+1)\times 1.3.5\ldots(2p+1)}{1.3.5\ldots(2m+2n+2p+3)}\cdot 4\pi abc.$$

When $m=n=p=0$, $V=\frac{4}{3}\pi abc$, volume of $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$.

When $m=n=p=1$, $V=\frac{4}{35}\pi abc$, volume of $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1$.

When $m=n=p=2$, $V=\frac{4.\,5\,\pi abc}{3.7.11.13}$, volume of $\left(\frac{x}{a}\right)^{\frac{2}{5}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}+\left(\frac{z}{c}\right)^{\frac{2}{5}}=1$.

When $m=n=p=3$, $V=\frac{4.5.7\pi abc}{9\ 11.13.17.19}$, volume of $\left(\frac{x}{a}\right)^{\frac{2}{7}}+\left(\frac{y}{b}\right)^{\frac{2}{7}}+\left(\frac{z}{c}\right)^{\frac{2}{7}}=1$

When $m=0$, $n=1$, $p=2$, $V=\frac{4}{21}\pi abc$, volume of $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{5}}=1$.

When $m=1$, $n=2$, $p=3$, $V=\frac{4\pi abc}{3.11.13}$, volume of $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}+\left(\frac{z}{c}\right)^{\frac{2}{7}}=1$.

Formulae (1), and (2) will do for any admissible values of m, n, p.

Let $m=n=p=\frac{3}{2}$; then $V=\frac{4abc\times 4.4.4}{12.\ 10}\cdot\frac{[\Gamma(2)]^3}{\Gamma(5)}=\frac{4}{45}\,abc$,

the volume of $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}}=1$.

When $m=n=\frac{3}{2}$, $A=ab\frac{4.\ 4}{4.\ 3}\cdot\frac{[\Gamma(2)]^2}{\Gamma(3)}=\frac{2}{3}ab$, the area of the

curve $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$.

The above formulae have been expressed in a little different form in the *Mathematical Magazien*, but they are so useful that they will bear repetion here.